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The characteristic polynomial of ladder digraph and an annihilating uniqueness theorem

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Abstract

Let G be a digraph with n vertices and $A(G)$ be its adjacency matrix. A monic polynomial $f(x)$ of degree at most n is called an annihilating polynomial of G if $f(A(G)) = 0$. G is said to be annihilatingly unique if it possesses a unique annihilating polynomial. In this paper, we give the explicit expression for the characteristic polynomial of the ladder digraph and show that the ladder digraph is annihilatingly unique.

1. Introduction

All graphs considered in this paper are directed, finite, loopless and without multiple arcs. Undefined terms and notations can be found in [1].

For a digraph G, two vertices are called adjacent if they are connected by an arc. The adjaceny matrix $A(G) = (a_{ij})$ of a digraph G with vertex set $\{1, 2, \dots, n\}$ is a square matrix of order n where the (i,j) entry, a_{ij}, is equal to the number of arcs starting at vertex i and terminating at the vertex j.

Let G be a digraph with n vertices and $A(G)$ its adjacency matrix. A monic polynomial $f(x)$ of degree at most n with $f(A(G)) = 0$ is called an annihilating polynomial of G. The existence of annihilating polynomial of G is guaranteed by its characteristic polynomial. G is said to be annihilatingly unique if it possesses a unique annihilating polynomial. Annihilating uniqueness of digraphs are first studied by Lam and Lim [2]. Dipaths and diwheels are examples of annihilatingly unique digraphs.

In the sequel, we need the following well-known results (see e.g. [1]).

Theorem 1.1. *The annihilating polynomial $f(x)$ of any digraph G with adjacency matrix $A(G)$ is unique if and only if $m(x) = \psi(x)$ where $m(x)$ is the minimum polynomial of $A(G)$ and $\psi(x)$ is the characteristic polynomial of $A(G)$.*

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Lemma 1.2. *Suppose that $A(G)$ is the adjacency matrix of a digraph G. Then the (i,j) entry of $A^r(G)$ is the number of different diwalks of length r from vertex i to vertex j.*

Theorem 1.3 (Cvetkovic et al. [1; Theorem 1.2, p. 32]). *Let $x^n + a_1x^{n-1} + \cdots + a_n$ be the characteristic polynomial of a digraph G. Then for every $i = 1, 2, \ldots, n$*

$$a_i = \sum(-1)^{P(L)},$$

where the sum is taken over all linear directed subgraphs (i.e. directed subgraphs with only dicycles as components) L of G with exactly i vertices; $P(L)$ is the number of components in L.

2. The ladder digraph $\vec{L}_{2l}$

The ladder digraph, denoted by $\vec{L}_{2l}$ (l a positive integer) is a digraph with vertex set $V = \{1, 2, \ldots, 2l\}$ together with the following arcs:

$$i \to i+1, \quad i = 1, 2, \ldots, 2l-1,$$

$$2j \to 2j-3, \quad j = 2, 3, \ldots, l.$$

The ladder digraph $\vec{L}_{2l}$ is illustrated in Fig. 1.

The ladder digraph $\vec{L}_{2l}$ possesses l rungs and l is called the level of the ladder digraph. Thus, the ladder digraph $\vec{L}_{2l}$ of l levels has $n = 2l$ vertices and $(3l - 2)$ arcs.

The only linear directed subgraphs of $\vec{L}_{2l}$ are the unions of disjoint copies of $\vec{C}_4$. From Theorem 1.3, the characteristic polynomial of $\vec{L}_{2l}$ is of the form

$$\begin{aligned}\psi(x) &= x^{2l} - a_4^{(l)}x^{2l-4} + a_8^{(l)}x^{2l-8} + \cdots + (-1)^r a_{4r}^{(l)}x^{2l-4r} \ldots \\ &= x^{2l} + \sum_{r=1}^{[l/2]} (-1)^r a_{4r}^{(l)}x^{2l-4r},\end{aligned} \tag{2.1}$$

where $[x]$ denotes the largest integer not bigger than x and $a_{4r}^{(l)}$ denotes the number of linear directed subgraphs of $\vec{L}_{2l}$ with exactly $4r$ vertices.

We note the following:

(i) When l is odd,

$$\begin{aligned}\psi(x) = x^{2l} &- a_4^{(l)}x^{2l-4} + a_8^{(l)}x^{2l-8} + \cdots \\ &+ (-1)^{[l/2]-1}a_{2l-6}^{(l)}x^6 + (-1)^{[l/2]}a_{2l-2}^{(l)}x^2.\end{aligned} \tag{2.2}$$

(ii) When l is even,

$$\begin{aligned}\psi(x) = x^{2l} &- a_4^{(l)}x^{2l-4} + a_8^{(l)}x^{2l-8} + \cdots \\ &+ (-1)^{l/2-1}a_{2l-4}^{(l)}x^4 + (-1)^{[l/2]}.\end{aligned} \tag{2.3}$$

$$a_0^{(l)} = 1, \quad l = 0, 1, 2, \ldots \tag{2.4}$$

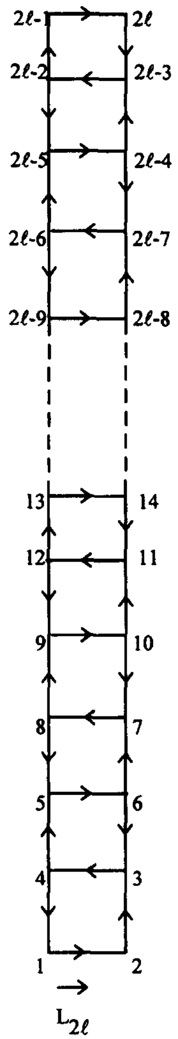


Fig. 1.

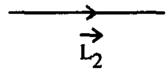


Fig. 2.

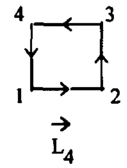


Fig. 3.

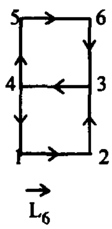


Fig. 4

$$\begin{aligned} a_{4r}^{(l)} &= 1 \quad \text{if } l = 2r \\ a_{4r}^{(l)} &= 0 \quad \text{if } l < 2r \text{ (or } n < 4r). \end{aligned} \tag{2.5}$$

The following are some examples of ladder digraphs.

Example 1. $\vec{L}_2$ as shown in Fig. 2 is a ditree $\vec{T}_2$ and $\psi(x) = x^2$.

Example 2. $\vec{L}_4$ shown in Fig. 3 is a dicycle $\vec{C}_4$, and has 2 levels. This ladder digraph consists of only 1 linear directed subgraph $\vec{C}_4$. Therefore, $a_4^{(2)} = 1$; hence $\psi(x) = x^4 - 1$.

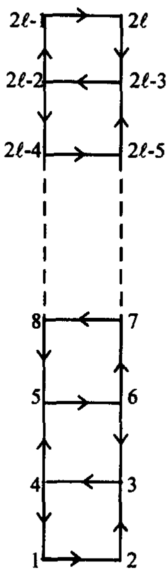


Fig. 5.

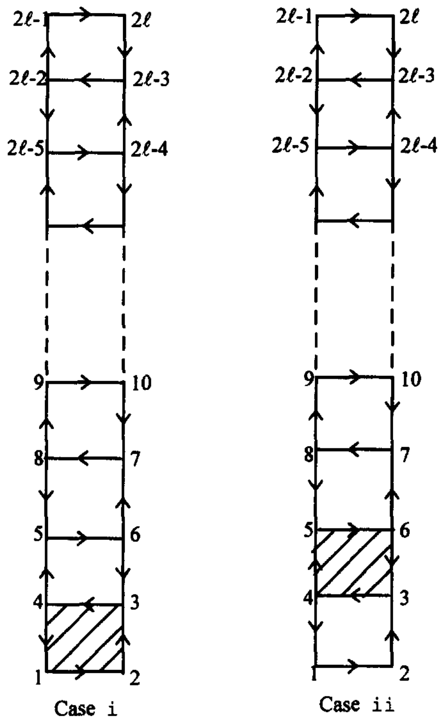


Fig. 6.

Example 3. The ladder digraph $\vec{L}_6$ as shown in Fig. 4 has 3 levels and consists of only 2 cycles of $\vec{C}_4$ as linear directed subgraphs. Hence,

$$a_4^{(3)} = 2, \qquad a_8^{(3)} = 0 \quad \text{since } 2l < 8.$$

Hence $\psi(x) = x^6 - 2x^2$.

We now proceed to obtain a recurrence relation for the coefficients $a_{4r}^{(l)}$ of the ladder digraph $\vec{L}_{2l}$.

Lemma 2.1. *Suppose $\vec{L}_{2l}$ is a ladder digraph with characteristic polynomial $\psi(x) = x^{2l} + \sum_{r=1}^{[l/2]}(-1)^r a_{4r}^{(l)} x^{2l-4r}$. Then*

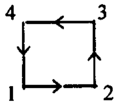
$$a_4^{(l)} = l - 1, \tag{2.6}$$

$$a_{4r}^{(l)} = a_{4r}^{(l-1)} + a_{4(r-1)}^{(l-2)}, \quad r = 2, 3, \ldots, [l/2]. \tag{2.7}$$

Proof. (a) As shown in Fig. 5, the number of linear directed subgraphs with exactly 4 vertices is $(l-1)$. Hence, $a_4^{(l)} = l - 1$.

(b) $a_{4r}^{(l)}$ is the number of linear directed subgraphs of $\vec{L}_{2l}$ with exactly $4r$ vertices. $a_{4r}^{(l)}$ can be computed as follows (see Fig. 6).

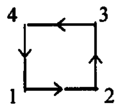
Case (i). The cycle



is included in the computation.

It suffices to get $(r-1)$ disjoint copies of $\vec{C}_4$ in $\vec{L}_{2(l-2)}$, amounting to obtain the number of linear directed subgraphs of $\vec{L}_{2(l-2)}$ with exactly $4(r-1)$ vertices. This is precisely $a_{4(r-1)}^{(l-2)}$.

Case (ii). The cycle



is not included in the computation. This is actually the number of getting r disjoint copies of $\vec{C}_4$ in $\vec{L}_{2(l-1)}$, which is the number of linear directed subgraphs of $\vec{L}_{2(l-1)}$ with exactly $4r$ vertices, i.e. $a_{4r}^{(l-1)}$.

Hence,

$$a_{4r}^{(l)} = a_{4(r-1)}^{(l-2)} + a_{4r}^{(l-1)}, \quad r = 2, 3, \ldots, [l/2].$$

Making use of Lemma 2.1, we can compute the characteristic polynomial of $\vec{L}_{2l}$ inductively. We list in Table 1 the first few characteristic polynomials of $\vec{L}_{2l}$ for $l = 1, 2, \ldots, 12$.

We observe from expressions (2.4), (2.5) that when $r = 1$, expression (2.7) also includes expression (2.6),

$$\begin{aligned} a_4^{(l)} &= a_4^{(l-1)} + a_0^{(l-2)} \\ &= (l-2) + 1 = l - 1. \end{aligned}$$

Table 1

Level, l	n	$\psi(x)$
1	2	x^2
2	4	$x^4 - 1$
3	6	$x^6 - 2x^2$
4	8	$x^8 - 3x^4 + 1$
5	10	$x^{10} - 4x^6 + 3x^2$
6	12	$x^{12} - 5x^8 + 6x^4 - 1$
7	14	$x^{14} - 6x^{10} + 10x^6 - 4x^2$
8	16	$x^{16} - 7x^{12} + 15x^8 - 10x^4 + 1$
9	18	$x^{18} - 8x^{14} + 21x^{10} - 20x^6 + 5x^2$
10	20	$x^{20} - 9x^{16} + 28x^{12} - 35x^8 + 15x^4 - 1$
11	22	$x^{22} - 10x^{18} + 36x^{14} - 56x^{10} + 35x^6 - 6x^2$
12	24	$x^{24} - 11x^{20} + 45x^{16} - 84x^{12} + 70x^8 - 21x^4 + 1$

Expression (2.7), $a_{4r}^{(l)} = a_{4r}^{(l-1)} + a_{4(r-1)}^{(l-2)}$ is in fact a difference equation of 2 variables l and r subject to the conditions

(i) $a_0^{(l)} = 1$, $l = 0, 1, \ldots,$
(ii) $a_{4r}^{(l)} = 1$, $l = 2r$,
(iii) $a_{4r}^{(l)} = 0$, $l < 2r$.

It can easily be shown that $a_{4r}^{(l)} = \binom{l-r}{r}$ is the solution to the difference equation $a_{4r}^{(l)} = a_{4r}^{(l-1)} + a_{4(r-1)}^{(l-2)}$, $r = 2, 3, \ldots, [l/2]$. Hence, we obtain the following theorem. $\square$

Theorem 2.2. *The characteristic polynomial of the ladder digraph $\vec{L}_{2l}$ is given by*

$$\psi(x) = x^{2l} + \sum_{r=1}^{[l/2]} (-1)^r \binom{l-r}{r} x^{2l-4r}.$$

Next, we proceed to show that the ladder digraphs are annihilatingly unique. We prove a more general result.

Theorem 2.3. *Let $\vec{Q}$ be a digraph with vertex set $\{1, 2, \ldots, n\}$ together with the arcs $(1,2), (2,3), \ldots, (n-1, n)$ and any number of additional arcs from vertex j to vertex i where $1 \leqslant i < j \leqslant n$.*

Then $\vec{Q}$ is annihilatingly unique.

Proof. Let $0 < k < n$ and

$$g(x) = x^k + \sum_{i=1}^{k} a_i x^{k-i}$$

with arbitrary coefficients a_i.

In $\vec{Q}$, there is precisely one diwalk of length k, and no diwalk of length smaller than k, from vertex 1 to vertex $k+1$.

Thus,

$$[A^k(\vec{Q})]_{1,k+1} = 1$$

and

$$[A^i(\vec{Q})]_{1,k+1} = 0$$

for $i = 0, 1, \ldots, k-1$.

Hence, $[g(A(\vec{Q})]_{1,k+1} = 1$; consequently,

$$g(A(\vec{Q})) \neq 0.$$

Therefore, $\vec{Q}$ is annihilatingly unique. $\square$

As a special case, we have the following result.

Theorem 2.4. *For any positive integer l, the ladder digraph $\vec{L}_{2l}$ with l levels is annihilatingly unique.*

References

[1] D.M. Cvetkovic, M. Doob and H. Sachs, Spectra of Graphs (Academic Press, New York, 1980).

[2] K.S. Lam and C.K. Lim, Annihilating uniqueness of q-diwheels, in: K.P. Shum, C.C. Yang and Yang Le, eds., Proc. 1st Asian Mathematical Conf., 1990, Hong Kong (World Scientific, Singapore, 1990) 267–270.